No Belief without Reason: Logics of Argument-Based Beliefs

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Abstract

The paper addresses how the information state of an agent relates to the arguments that the agent endorses. Information states are modeled in doxastic logic and arguments by recasting abstract argumentation theory in a modal logic format. The two perspectives are combined by an application of the theory of product logics, delivering sound and complete systems in which the interaction of arguments and beliefs is investigated.

1 Introduction

With epistemic logic we mean logics for knowledge and belief, i.e., logics that describe the information state of an agent by the alternatives it considers possible. Abstract argumentation [Dung, 1995], on the other hand, describes the information state of an agent by the set of arguments the agent 'endorses' or holds as justified.

The paper combines these two dimensions in a unified framework where questions concerning the interaction of arguments and beliefs can be systematically addressed, such as: Is the set of arguments supporting an agent's doxastic state 'justifiable' from the point of view of abstract argumentation (e.g., conflict-free, admissible, stable, etc.)?

Technically, the combination of the two perspectives—epistemic and argumentation-theoretic—is achieved by deploying techniques and results from the theory of product modal logics [Gabbay *et al.*, 2003]. The key idea consists in taking states in a doxastic (Kripke) model and arguments in a (Dung) attack graph as two orthogonal dimensions for the description of the information state of an agent. This intuition suggests the use of bi-dimensional structures for the study of argument-based beliefs. The logics obtained are studied with respect to their axiomatization and finite model property.

More generally, the paper lays a bridge between epistemic logic and argumentation theory. We hope that the results presented can foster further interaction between the two fields.

Related work To the best of our knowledge, the only works to date attempting to interface argumentation with epistemic logic are [Grossi, 2012] and [Schwarzentruber *et al.*, 2012]. The first is concerned with the analysis, in dynamic epistemic logic [van Ditmarsch *et al.*, 2007], of the fixpoint behavior of some argumentation theoretic notions, and the second enriches the standard framework of abstract argumentation by enabling arguers to hold beliefs about other arguers' available arguments.

In its broad purposes, the present paper can probably be better related to recent work (in particular, [Artemov, 2008; van Benthem and Pacuit, 2011]) aiming at explicitly modeling the 'justifications' or 'reasons' upon which agents happen to base their information state. Our paper shows the viability of using product logics for this type of analysis.

Outline of the paper Section 2 prepares the ground recapitulating some basic notions from both epistemic logic and abstract argumentation. We then proceed in a modular fashion. Section 3 introduces a logic that combines the simplest modal logic of argumentation (logic K, cf. [Grossi, 2010]) and the simplest modal doxastic logic ($\mathsf{KD45}$). First argument-belief interaction properties are formalized and discussed. Section 4 introduces and studies a more expressive logic, able to formalize: ample fragments of Dung's argumentation theory and a rich set of doxastic attitudes based on the set of arguments that an agent endorses. Section 5 concludes pointing at some future research directions.

2 Preliminaries

We start by introducing the basic building blocks of our analysis: simple structures for representing beliefs and for representing arguments and their attacks. We then move to motivate a specific way of combining the two: products. Although we give all necessary definitions, space limitations demand we do this succinctly: the reader may wish to consult [Meyer and van der Hoek, 1995] for more background on doxastic logic, [Baroni and Giacomin, 2009; Baroni *et al.*, 2011] for abstract argumentation, and [Gabbay and Shehtman, 1998] for products in modal logic.

2.1 Doxastic structures

Definition 1 (Doxastic frame)**.** *A doxastic frame is a tuple $\mathcal{D} = \langle S, B \rangle$ where: i) S is a non-empty set of states; ii) $B \subseteq S$, s.t., $B \neq \emptyset$. The class of all doxastic frames is denoted $\mathfrak{D}$.*

A doxastic frame represents the non-empty set B of states that an agent holds as most plausible among the set of all states S. We will refer to elements of B as *doxastic alternatives* or *doxastically accessible states*.

Given a doxastic frame, a Kripke model is obtained by adding a valuation function $\mathcal{V} : \mathbf{P} \longrightarrow \wp(S)$ interpreting a set of atoms $\mathbf{P}$. A doxastic modality $\diamondminus_{\mathsf{B}}$ can then be interpreted as follows:

$$\langle \mathcal{D}, \mathcal{V} \rangle, s \models \diamondminus_{\mathsf{B}} \varphi \iff \exists s' \in B : \langle \mathcal{D}, \mathcal{V} \rangle, s' \models \varphi \quad (1)$$

For any diamond $\Diamond$ in this paper, we define an associated $\Box$ as $\Box\varphi = \neg \Diamond \neg\varphi$. So $\boxminus_{\mathsf{B}}\varphi$ means that the agent believes φ.

Doxastic frames are somewhat simpler than the structures typically used to study beliefs, namely relational frames where the accessibility relation is transitive and euclidean (cf. [Meyer and van der Hoek, 1995]). The two, however, can be proven equivalent for the purpose of this paper, using some standard modal logic arguments:

Theorem 1. *On the basic modal language, the class of doxastic frames is modally equivalent to the class of transitive and euclidean relational frames.*

Proof. Let $\mathfrak{F}$ the class of transitive and euclidian frames and $\mathfrak{F}_g$ the class of point-generated transitive and euclidean frames. We know that $\mathfrak{F}$ and $\mathfrak{F}_g$ are modally equivalent [Blackburn *et al.*, 2001]. [LEFT TO RIGHT] Assume $\mathfrak{D} \models \varphi$. Let $\mathcal{F}_s = \langle S_s, R_s \rangle \in \mathfrak{F}_g$ and $s \in S_s$. Define $B = \{x \mid sR_sx\}$ and notice that $\mathcal{D}_s = \langle S_s, B \rangle \in \mathfrak{D}$ and hence $\mathcal{D}_s \models \varphi$. An easy induction shows that $\mathcal{D}_s$ and $\mathcal{F}_s$ are modally equivalent and thus $\mathfrak{F} \models \varphi$ [RIGHT TO LEFT] Assume $\mathfrak{F} \models \varphi$. Let $\mathcal{D} = \langle S, B \rangle \in \mathfrak{D}$. Define a relation R on S by xRy iff $y \in B$. It is easy to see that R is transitive and euclidean and hence $\mathcal{F} = \langle S, R \rangle \in \mathfrak{F}$. An easy induction proves that $\mathcal{D}$ and $\mathcal{F}$ are modally equivalent, from which $\mathfrak{D} \models \varphi$. $\Box$

So the logic of the class of doxastic frames is completely axiomatized by the standard axiom system for logic $\mathsf{KD45}$ containing: the rules Modus Ponens ($\mathbf{MP}$: form φ and $\varphi \to \psi$, infer ψ) and Necessitation ($\mathbf{Nec}$: from φ, infer $\boxminus_{\mathsf{B}}\varphi$); the axioms $\mathbf{K} : \boxminus_{\mathsf{B}}(\varphi \to \psi) \to (\boxminus_{\mathsf{B}}\varphi \to \boxminus_{\mathsf{B}}\psi)$ representing the agent's ability to reason propositionally, $\mathbf{D} : \neg \boxminus_{\mathsf{B}} \bot$ (beliefs are consistent) and the axioms $\mathbf{4} : \boxminus_{\mathsf{B}}\varphi \to \boxminus_{\mathsf{B}} \boxminus_{\mathsf{B}} \varphi$ and $\mathbf{5} : \boxminus_{\mathsf{B}}\varphi \to \boxminus_{\mathsf{B}} \neg \boxminus_{\mathsf{B}} \varphi$ representing positive and negative introspection, respectively. For later reference, recall that $\mathsf{S5}$ is the logic $\mathsf{KD45} + \mathbf{T} : \boxminus_{\mathsf{B}}\varphi \to \varphi$.

2.2 Argumentative structures

We start by the key structure of abstract argumentation:

Definition 2 (Attack graphs [Dung, 1995])**.** *An attack graph is a tuple $\mathcal{A} = \langle A, \to \rangle$ where: i) A is a non-empty set of arguments; ii) $\to \subseteq A^2$ is a binary relation ($a \to b$ stands for a attacks b). The class of all attack graphs is denoted $\mathfrak{A}$.*

These relational structures are the building blocks of abstract argumentation theory. Once A is taken to represent a set of arguments, and $\to$ an 'attack' relation between arguments, the study of these structures provides general insights on how competing arguments interact, and structural properties of subsets of A can be taken to formalize how collections of arguments form 'justifiable' positions in an argumentation ([Baroni and Giacomin, 2009; Baroni *et al.*, 2011]).

In this paper we will touch upon the argumentation-theoretic notions of conflict-freeness, self-defense, admissibility, complete and stable extensions. Table 1 recapitulates these notions for the ease of the reader.

A Kripke model $\mathcal{M} = \langle A, \leftarrow, \mathcal{V} \rangle$ can be obtained from an attack graph by inverting the attack relation ($a \leftarrow b$ denotes that a is attacked by b) and by adding a valuation function $\mathcal{V} : \mathbf{P} \longrightarrow \wp(A)$ interpreting a set of propositional atoms $\mathbf{P}$. Consider now a modality $\diamondvert_{\mathsf{A}}$ with the following semantics:

$$\langle \mathcal{A}, \mathcal{V} \rangle, a \models \diamondvert_{\mathsf{A}} \varphi \iff \exists b \in A : a \leftarrow b \;\&\; \langle \mathcal{A}, \mathcal{V} \rangle, b \models \varphi \; (2)$$

An argument a satisfies $\diamondvert_{\mathsf{A}}\varphi$ iff some attacker b of a satisfies φ. The logic of $\diamondvert_{\mathsf{A}}$ defined by the class of attack graphs is, obviously, K. [Grossi, 2010] shows that modal logic K can express a number of argumentation theoretic notions from [Dung, 1995], such as: $\neg \diamondvert_{\mathsf{A}} p$, expressing that the current argument is not attacked by p; or $\boxvert_{\mathsf{A}} \diamondvert_{\mathsf{A}} p$ expressing that the current argument is 'defended' by p-arguments (i.e., its attackers are attacked by p-arguments). The logic K is axiomatized by rules $\mathbf{MP}$ and $\mathbf{Nec}$, and axiom $\mathbf{K}$.

2.3 Doxo-argumentative structures

Let us start with a simple motivating example:

Example 1 (After [Modgil, 2009])**.** *Consider two individuals exchanging arguments about the weather forecast. Argument a: "Today will be dry in London since the BBC forecast sunshine". And argument a': "Today will be wet in London since CNN forecast rain". We have two arguments (a and a') concerned with whether the 'real' situation is a state s where the sun shines in London or in a state s' where it rains in London.*

In general, starting from a set of doxastic alternatives S and a set of 'arguments' A, we are after structures that can support the analysis of how elements of S interact with elements of A. We want to be able to express properties of state-argument pairs (s, a) such as "a supports s" or "all doxastic states supported by this argument have property p".

Example 2 (After [Modgil, 2009], continued)**.** *Let then $S = \{s, s'\}$ and $A = \{a, a'\}$. We can represent the simple scenario of Example 1 by the model on the left of Figure 1, where the dark circles indicate that the pairs at issue have a property of interest (in this case the property of 'support'): a supports s and a' supports s'.*

So our domain becomes the Cartesian product $S \times A$. Now, if S and A also come equipped with accessibility relations—the doxastic one in case of S and the attack one in case of A—then studying how the logics of these relations interact in $S \times A$ would allow one to talk about properties of state-argument pairs that are of a doxastic and argumentation-theoretic type in the same language. The paper takes this perspective and sets out to develop a formal theory of how arguments and their attacks relate to the doxastic state of an agent. The key tool in accomplishing this, is that of product logics.

Conflict-free	$X \subseteq \{x \mid \nexists y \in X : x \leftarrow y\}$
Self-defended	$X \subseteq \{x \mid \forall y : x \leftarrow y \Rightarrow \exists z \in X, y \leftarrow z\}$
Admissible	X is conflict-free and self-defended
Complete extension	X is conflict-free & $X = \{x \mid \forall y : x \leftarrow y \Rightarrow \exists z \in X, y \leftarrow z\}$
Stable extension	$X = \{x \mid \nexists y \in X : x \leftarrow y\}$

Table 1: Properties of a set X in $\mathcal{A} = \langle A, \to \rangle$.

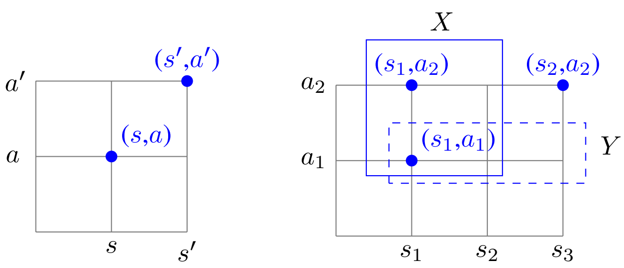

Figure 1: Two doxo-argumentative structures.

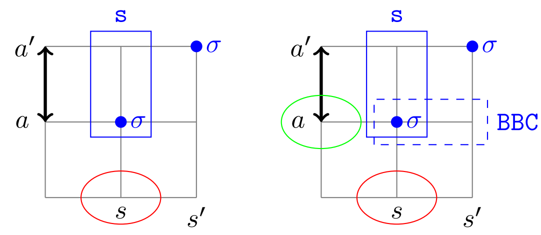

Figure 2: Rendering of Example 1 as a $\mathsf{KD45} \times \mathsf{K}$ model (left) and a DA model (right). Universal relations are not depicted.

Remark 1 (Properties of states, arguments and their pairs)**.** *The above set up allows one to represent any property of state-argument pairs. For instance, with respect to the support relation, one can represent cases where the same argument supports several states or where a same state is supported by several arguments.*

Importantly, the set up allows one to represent properties of states-only, or arguments-only. For instance, to express that a state s has a property X (e.g., 'sunshine in London') it suffices for X to be true of all the state-argument pairs whose state is s, i.e., to be a set of columns in the Cartesian plane. Similarly, properties Y of arguments alone (e.g., 'being upheld by BBC') can be represented in the same fashion, i.e., by sets of rows of the Cartesian plane.

Figure 1 (right) illustrates all these different properties. Dark circles indicate a relation (e.g., of support) between arguments and states: a_2 supports both s_1 and s_3; s_1 is supported by both a_1 and a_2. Rectangle X represents a state-only property, of states s_1 and s_2, and rectangle Y (dashed line) represents an argument-only property, of argument a_1.

2.4 Product logics

The product of two (uni-)modal logics[1] is defined as follows [Gabbay and Shehtman, 1998]. The product $\mathcal{F} \times \mathcal{F}'$ between two frames $\mathcal{F} = \langle S, R\rangle$ and $\mathcal{F}' = \langle S', R'\rangle$ is the frame $\langle S \times S', H, V\rangle$ where:

$$
\begin{aligned}
(s, s')H(t, t') &\iff sRt \text{ AND } s' = t' \\
(s, s')V(t, t') &\iff s'R't' \text{ AND } s = t
\end{aligned}
$$

Intuitively, the product of two frames can be depicted as a Cartesian plane where H is the relation on the 'horizontal' dimension consisting of the set S and V is the relation on the the 'vertical' one consisting of set S'. Following [Marx, 1999], we will use $\diamondminus$ to denote the modality interpreted over H—'horizontal' modality—and $\diamondvert$ the modality interpreted over V—'vertical' modality.

The product of two classes of frames $\mathfrak{F}$ and $\mathfrak{F}'$ is $\{\mathcal{F} \times \mathcal{F}' \mid \mathcal{F} \in \mathfrak{F} \text{ AND } \mathcal{F}' \in \mathfrak{F}'\}$. Now, given two logics L and L' the product $\mathsf{L} \times \mathsf{L}'$ is the logic of the class of frames defined by the product of the two largest classes $\mathfrak{F}$ and $\mathfrak{F}'$ for which the two logics are complete. For instance, $\mathsf{K} \times \mathsf{K}$ is the logic of the class of all frames consisting of the product of two frames. Here we study products between logics L and L' where L—the 'horizontal' logic—is a doxastic logic and L'—the 'vertical' logic—is a modal logic for argumentation.

[1] The multi-modal case is a straightforward generalization. Cf. [Gabbay and Shehtman, 1998].

3 A simple product logic: $\mathsf{KD45} \times \mathsf{K}$

As a first framework in which to investigate interaction principles between arguments and doxastic states we consider the product of the simplest doxastic logic, namely $\mathsf{KD45}$, with the simplest modal logic of attack graphs, i.e., K.

3.1 Syntax and semantics

The language $\mathcal{L}(\diamondminus_{\mathsf{B}}, \diamondvert_{\mathsf{A}})$, has the following BNF:

$$p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \diamondminus_{\mathsf{B}}\varphi \mid \diamondvert_{\mathsf{A}}\varphi$$

where p belongs to the set of atoms $\mathbf{P}$. For any language $\mathcal{L}$ that we consider, the variant $\mathcal{L}^{\sigma}$ adds an atom σ to it, where σ intuitively says of (s, a) the s is supported by a. Semantics is given as follows. Let $\mathcal{D}$ be a doxastic frame on S and $\mathcal{A}$ an attack graph on A. A $\mathsf{KD45} \times \mathsf{K}$ model is a structure $\mathcal{M} = \langle \mathcal{D} \times \mathcal{A}, \mathcal{V}\rangle$ where $\mathcal{V} : \mathbf{P} \longrightarrow \wp(S \times A)$. The satisfaction relation is defined by the standard Boolean clauses plus the following clauses (cf. Formulae (1) and (2)):

$$
\begin{aligned}
\mathcal{M}, (s, a) \models \diamondvert_{\mathsf{A}}\varphi &\iff \exists a' \in A : a \leftarrow a' \;\&\; \mathcal{M}, (s, a') \models \varphi \\
\mathcal{M}, (s, a) \models \diamondminus_{\mathsf{B}}\varphi &\iff \exists s' \in B : \mathcal{M}, (s', a) \models \varphi
\end{aligned}
$$

The claim $\mathcal{M}, (s, a) \models \varphi$ can be interpreted as: given the 'actual' state is s and the 'currently entertained' argument is a, φ holds. So $\boxminus_{\mathsf{B}}\varphi$ expresses the property that, by keeping fixed the current argument, all pairs consisting of the current argument and a doxastically accessible state, satisfy φ. Intuitively: it is believed that φ holds of the current argument. Similarly, modalities $\diamondvert_{\mathsf{A}}$ and $\boxbar_{\mathsf{A}}$ express properties of the attack relation. So $\diamondvert_{\mathsf{A}}\varphi$ expresses the property that, by keeping the current state fixed, there exists a pair consisting of the current state and an argument that attacks the current argument, and this pair satisfies φ.

Remark 2 (Satisfaction in products)**.** *As usual, formulae are interpreted on pointed models "$\mathcal{M}, (s, a)$". So, when we interpret a formula we fix both an argument and a state and $\mathcal{M}, (s, a) \models \varphi$ can be interpreted as: given the 'actual' state is s and the 'currently entertained' argument is a, φ holds.*

Example 3 (After [Modgil, 2009], continued)**.** *We extend Example 1 by making explicit that the two arguments a and a' attack one another, and that the agent believes the actual state is s (the one supported by argument a), so that the set of doxastic alternatives B is $\{s\}$ (left of Figure 2, where the ellipsis* *encloses the set of doxastic alternatives). Dark circles denote the truth set of atom σ (representing 'support') and the rectangle denotes the truth set of an atom s (for 'sunshine'). Notice that s is here a 'column' property (Remark 1). Arrows on the vertical dimension denote attack.*

Here are formulae true at (s, a): (i) $\boxminus_{\mathsf{B}}\sigma$; (ii) $\boxminus_{\mathsf{B}}(\sigma \wedge \boxbar_{\mathsf{A}}\neg\sigma)$. Intuitively: (i) says that I believe the current argument is supportive, that is, all my doxastic alternatives are supported by the current argument; (ii) says that all my doxastic alternatives are supported by the current argument and all its attackers are not supportive of my doxastic alternatives. Some of the formulae true at (s', a'): (i) $\neg\mathsf{s} \wedge \boxminus_{\mathsf{B}}\mathsf{s}$; (ii) $\boxminus_{\mathsf{B}}\neg\sigma$; (iii) $\boxminus_{\mathsf{B}} \Diamond_{\mathsf{A}} \sigma$. Intuitively: (i) expresses a standard false belief property 'I believe s of the current argument but s is false'; (ii) expresses that I believe the current argument is not supportive, that is, no doxastic alternative is supported by the current argument; (iii) states that for all doxastic alternatives, the current argument is attacked by an argument supporting that alternative.

3.2 Metalogical results

It is worth now giving a brief technical overview of $\mathsf{KD45}\times\mathsf{K}$. Regarding an axiomatisation, logic $\mathsf{KD45} \times \mathsf{K}$ is the logic on $\mathcal{L}$ of the class of frames consisting of the product of a single-agent doxastic frame and an attack frame. It is axiomatized by taking $\mathsf{KD45}$ for $\Diamond_{\mathsf{B}}$, and K for $\Diamond_{\mathsf{A}}$, plus the two following axioms:

Com	$\Diamond_{\mathsf{A}} \Diamond_{\mathsf{B}} \varphi \leftrightarrow \Diamond_{\mathsf{B}} \Diamond_{\mathsf{A}} \varphi$
Con	$\Diamond_{\mathsf{B}} \boxbar_{\mathsf{A}} \varphi \rightarrow \boxbar_{\mathsf{A}} \Diamond_{\mathsf{B}} \varphi$

We will come back later to the intuitive meaning of these axioms in our context. The completeness of this axiom system is established as a corollary of known general theorems [Gabbay and Shehtman, 1998, Theorem 7.12] or [Gabbay *et al.*, 2003, Theorem 5.9]: we only need to check that the axioms for K and $\mathsf{KD45}$ are either without atoms (i.e., frame formulae), or else have a specific syntactic form (called *pseudo-transitivity*), which is the case.[2]

A logic has the *product finite model property* w.r.t. class $\mathfrak{F}\times\mathfrak{F}'$ iff every satisfiable formula on that class can be satisfied on a model built on the product of finite frames in $\mathfrak{F}$ and $\mathfrak{F}'$.

Logic $\mathsf{KD45}\times\mathsf{K}$ has the (strong) product finite model property as every φ can be satisfied on a finite model of size exponential in the length of φ [Gabbay *et al.*, 2003, Theorem 6.56]. Logic $\mathsf{KD45} \times \mathsf{K}$ is therefore decidable and its satisfiability problem is NEXPTIME-complete [Gabbay *et al.*, 2003, Theorem 6.57].

3.3 Interaction of attacks and beliefs in $\mathsf{KD45} \times \mathsf{K}$

We now turn to the sort of insights that we gain by modeling the interaction of doxastic structures (Kripke frames) and argument structures (attack graphs) as a product, and what logic $\mathsf{KD45} \times \mathsf{K}$ allows us to say about such interaction.

Example 3 has already shown interaction properties expressible in $\mathsf{KD45} \times \mathsf{K}$. Here are a few more examples:

$$(a)\quad \boxbar_{\mathsf{A}}\Diamond_{\mathsf{A}}\top \qquad (b)\quad \sigma \wedge \Diamond_{\mathsf{A}}\top \wedge \boxbar_{\mathsf{A}}\sigma$$
$$(c)\quad \Diamond_{\mathsf{B}}\sigma \qquad (d)\quad \sigma \wedge \boxbar_{\mathsf{A}}\neg\Diamond_{\mathsf{B}}\sigma$$

To understand these properties, let us see what they express once evaluated at a pointed model $\mathcal{M}, (s, a)$. Formula (a) expresses that all attackers of a have an attacker, i.e., a is *defended* by some argument. Formula (c) expresses that all doxastically accessible alternatives are supported by the current argument and (b) states that the current argument a supports the current alternative s, but a has attackers and indeed all attackers of a also support s. Intuitively, a is therefore a 'weak' argument for supporting s. Formula (d) states again that the current argument supports the current alternative and that all attackers of the current argument do not support any alternative. This property expresses therefore a form of *safety* of a state-argument pair: the argument supports the alternative but there is no other alternative which is supported by an argument attacking the current one. Notice that this property weakens a purely argumentative notion of safety as inexistence of attackers ($\boxbar_{\mathsf{A}}\bot$), by requiring that if there are attackers, then these are not effective in supporting any doxastic alternative. The formula $\boxminus_{\mathsf{B}}(\sigma \rightarrow \boxbar_{\mathsf{A}}\neg \Diamond_{\mathsf{B}} \sigma)$ would then express the agent's belief of safety of the current argument, across the set of doxastic alternatives.

Let us move on to the interpretation of formulae **Com** and **Con**, using the support atom σ as a state-argument pair example property. Intuitively, $\Diamond_{\mathsf{A}} \Diamond_{\mathsf{B}} \sigma \leftrightarrow \Diamond_{\mathsf{B}} \Diamond_{\mathsf{A}} \sigma$ states that the property 'there is an attacker of the argument I'm currently entertaining, and a state I consider doxastically possible, such that the first supports the latter' can be formulated independently of the order of the diamonds involved. As to $\Diamond_{\mathsf{B}} \boxbar_{\mathsf{A}} \sigma \rightarrow \boxbar_{\mathsf{A}} \Diamond_{\mathsf{B}} \sigma$, it expresses that, if I hold a state as doxastically possible which is supported by all the attackers of an argument I currently entertain, then all the attackers of the argument I currently entertain support a doxastic alternative.

We list now validities of $\mathsf{KD45}\times\mathsf{K}$. It is worth reading them as interaction properties of arguments and doxastic states.

Proposition 1. *The following are validities of* $\mathsf{KD45} \times \mathsf{K}$*:*

$$\boxminus_{\mathsf{B}}(\Diamond_{\mathsf{A}}p \rightarrow \Diamond_{\mathsf{A}} \Diamond_{\mathsf{B}}\, p) \qquad \Diamond_{\mathsf{A}} \Diamond_{\mathsf{B}}\, p \rightarrow \boxminus_{\mathsf{B}} \Diamond_{\mathsf{A}} \Diamond_{\mathsf{B}}p$$
$$\Diamond_{\mathsf{A}} \boxminus_{\mathsf{B}}\, p \rightarrow \boxminus_{\mathsf{B}} \boxminus_{\mathsf{B}} \Diamond_{\mathsf{A}}p \qquad \Diamond_{\mathsf{B}} \boxbar_{\mathsf{A}}\, p \rightarrow \boxminus_{\mathsf{B}} \boxbar_{\mathsf{A}} \Diamond_{\mathsf{B}}p$$

Recall that p can be any property of a state-argument pair.

4 Believing & endorsing

We now want to express properties about the interaction of a given set of doxastic alternatives and a given set of entertained or 'endorsed' arguments. This is not possible in $\mathsf{KD45}\times\mathsf{K}$. To do this, we have to move from simple attack graphs $\langle A, \rightarrow\rangle$ to attack graphs isolating a designated non-empty set E of 'endorsed arguments': $\mathcal{E} = \langle A, \rightarrow, E\rangle$. We call them *enriched attack graphs* and we denote their class by $\mathfrak{E}$. We then proceed as above studying the logic of the products of doxastic frames with enriched attack graphs.

[2] More precisely, the theorem states that the product $\mathsf{L}_H \times \mathsf{L}_V$ of two logics L_H and L_V whose axioms are either formulae from the frame language (i.e., without atoms, like $\Diamond\top$) or have the form $\triangledown \Box p \rightarrow \triangle p$ where $\triangledown$ is a sequence of possibly different diamonds and $\triangle$ a sequence of possibly different boxes (so-called *pseudo-transitive formulae*), is completely axiomatized by the axioms of L_H, the axioms of L_V, plus the **Com** and **Con** axioms for each pairs of modalities in the combined language. Notice that pseudo-transitive formulae are Sahlqvist formulae.

4.1 Syntax and semantics

The language $\mathcal{L}^E$ is defined by the following BNF:

$$p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \diamondminus_{\mathsf{B}}\varphi \mid \diamondbar_{\mathsf{E}}\varphi \mid \diamondbar_{\mathsf{A}}\varphi \mid \diamondminus_{\mathsf{U}}\varphi \mid \diamondbar_{\mathsf{U}}\varphi$$

with $p \in \mathbf{P}$. As before, $\mathcal{L}^{E,\sigma}$ extends $\mathcal{L}^E$ with the designated atom σ. Modalities $\diamondminus_{\mathsf{B}}$ and $\diamondbar_{\mathsf{A}}$ are as above. As to the others: $\diamondbar_{\mathsf{E}}$ means 'for some endorsed argument by keeping fixed the current state'; $\diamondminus_{\mathsf{U}}$ means 'for some state by keeping fixed the current argument'; $\diamondbar_{\mathsf{U}}$ means 'for some argument by keeping fixed the current state'. Notice that $\diamondminus_{\mathsf{U}}$ and $\diamondbar_{\mathsf{U}}$ are nothing but universal modalities for the horizontal and, respectively, vertical dimensions. We refer to the fragment of $\mathcal{L}^E$ containing only $\diamondminus_{\mathsf{B}}$, $\diamondminus_{\mathsf{U}}$ modalities as its *horizontal fragment*, and to the fragment of $\mathcal{L}^E$ containing only $\diamondbar_{\mathsf{E}}$, $\diamondbar_{\mathsf{A}}$ and $\diamondbar_{\mathsf{U}}$ modalities as its *vertical fragment*.

The semantics for $\mathcal{L}^E$ is defined as follows. Let $\mathcal{F}$ be a doxastic frame on S and $\mathcal{A}$ an enriched attack graph on A. A model is a structure $\mathcal{M} = \langle \mathcal{D} \times \mathcal{E}, \mathcal{V} \rangle$ where $\mathcal{V} : \mathbf{P} \longrightarrow \wp(S \times A)$. The satisfaction relation is defined as follows (clauses are limited to the newly introduced operators):

$$\begin{array}{lcl}
\mathcal{M}, (s,a) \models \diamondbar_{\mathsf{E}}\varphi & \Longleftrightarrow & \exists a' \in E : \mathcal{M}, (s,a') \models \varphi \\
\mathcal{M}, (s,a) \models \diamondminus_{\mathsf{U}}\varphi & \Longleftrightarrow & \exists s' \in S : \mathcal{M}, (s',a) \models \varphi \\
\mathcal{M}, (s,a) \models \diamondbar_{\mathsf{U}}\varphi & \Longleftrightarrow & \exists a' \in A : \mathcal{M}, (s,a') \models \varphi
\end{array}$$

We call the logic on $\mathcal{L}^E$ defined by the class of the above models DA *doxastic argument logic*.

Remark 3 ('Column' and 'row' properties)**.** *Modalities $\diamondminus_{\mathsf{U}}$ and $\diamondbar_{\mathsf{U}}$ make it possilbe to express properties of states or arguments-only (cf. Remark 1). A state-only or 'column' property is expressed by $\boxbar_{\mathsf{U}}\varphi$ (i.e., φ holds of the current pair independently of the argument) and an argument-only or 'row' property is expressed by $\boxminus_{\mathsf{U}}\varphi$ (i.e., φ holds of the current pair independently of the state). Examples are, in the right model of Figure 2: $\boxbar_{\mathsf{U}}\mathsf{s}$ of state s, and $\boxminus_{\mathsf{U}}\mathsf{BBC}$ of argument a. We can then express that the agent* believes *φ, in the sense that all its doxastc alternatives have column property φ by $\boxminus_{\mathsf{B}} \boxbar_{\mathsf{U}} \varphi$ (in the example: $\boxminus_{\mathsf{B}} \boxbar_{\mathsf{U}} \mathsf{s}$). Similarly, we can express that all the arguments endorsed by the agent all have row property φ by $\boxbar_{\mathsf{E}} \boxbar_{\mathsf{U}} \varphi$ (in the example: $\boxbar_{\mathsf{E}} \boxminus_{\mathsf{U}} \mathsf{BBC}$).*

Example 4 (After [Modgil, 2009], continued)**.** *We expand Example 3 by casting it as a* DA *model and making thus explicit that the agent endorses argument a, so that $E = \{a\}$. The new model is depicted on the right of Figure 2 where the ellipsis on the vertical axis encloses the set of endorsed arguments. The following are validities of the above model (thus independent of the point of evaluation):*

$$\begin{array}{llll}
(a) & \boxminus_{\mathsf{B}} \diamondbar_{\mathsf{E}}\, \sigma & (b) & \boxbar_{\mathsf{E}} \boxminus_{\mathsf{B}} (\sigma \to \neg \diamondbar_{\mathsf{A}} \diamondminus_{\mathsf{B}}\sigma) \\
(c) & \boxminus_{\mathsf{B}} \boxbar_{\mathsf{U}}\, \mathsf{s} & (d) & \boxbar_{\mathsf{U}} \boxminus_{\mathsf{U}} ((\sigma \wedge \mathsf{s}) \to \neg \diamondbar_{\mathsf{A}} \diamondminus_{\mathsf{U}}(\sigma \wedge \mathsf{s}))
\end{array}$$

Intuitively: (a) expresses that the agent's beliefs are supported by arguments it endorses—it formalizes the motto "no belief without reason" in the title of the paper; (b) that for all pairs of endorsed arguments and doxastic alternatives, if the argument supports the alternative, then there is no attacker of that argument which supports some other alternative; (c) that all doxastic alternatives satisfy s*—i.e., the agent believes* s *(cf. Remark 3); (d) that the state-argument pairs satisfying $\sigma \wedge \mathsf{s}$ do not contain any attack between their arguments (cf. the formalization of conflict-freeness of sets of arguments in Section 4.3). Notice that none of these properties was expressible in the simpler language $\mathcal{L}(\diamondminus_{\mathsf{B}}, \diamondbar_{\mathsf{A}})$ of Section 3.*

Remark 4 (On the 'support' relation)**.** *In our set up, we have taken a liberal view on the notion of support of a state by an argument. We have seen support as just one of the possible relations between argument and states (others can be 'rejection', 'incompatibility', 'weak support', etc.) and modeled it through a dedicated atom σ, whose interpretation has not been constrained. However, meaningful classes of* DA *models can be isolated by strengthening our axiom system with axioms enforcing desirable properties like: $\diamondminus_{\mathsf{U}}\sigma$ (every argument supports some state); $\boxminus_{\mathsf{B}} \diamondbar_{\mathsf{E}}\, \sigma$ (the agent considers possible only states supported by some argument).*

4.2 Metalogical results

Horizontal logic: axiomatics

Let us first concern ourselves with axiomatizing the logic determined by the class of doxastic frames on the horizontal fragment of $\mathcal{L}^E$. Consider the logic—call it DA^H—defined by the rules and axioms of KD45 for modality $\diamondminus_{\mathsf{B}}$, the rules and axioms of S5 for modality $\diamondminus_{\mathsf{U}}$ plus:

$\mathbf{Inc}_{\mathsf{BU}}$	$\diamondminus_{\mathsf{B}}\varphi \to \diamondminus_{\mathsf{U}}\varphi$	$\mathbf{4}_{\mathsf{BU}}$	$\boxminus_{\mathsf{B}}\varphi \to \boxminus_{\mathsf{U}} \boxminus_{\mathsf{B}}\, \varphi$
$\mathbf{5}_{\mathsf{BU}}$	$\diamondminus_{\mathsf{B}}\varphi \to \boxminus_{\mathsf{U}} \diamondminus_{\mathsf{B}}\, \varphi$		

Lemma 1. DA^H *is sound and complete for class $\mathfrak{D}$.*

Proof sketch. Soundness is straightforward. As to completeness, the axiom system can be shown to be complete with respect to the class of frames $\mathcal{F} = \langle S, R_{\mathsf{B}}, R_{\mathsf{U}} \rangle$ consisting of one equivalence relation R_{U} (axiomatized by S5) which, within each of its equivalence classes, contains (containment is enforced by $\mathbf{Inc}_{\mathsf{BU}}$) a transitive and euclidean relation R_{B} (axiomatized by KD45) with the additional property (enforced by axioms $\mathbf{4}_{\mathsf{BU}}$ and $\mathbf{5}_{\mathsf{BU}}$) that within each equivalence class all states have access to the same set of states: $\forall x, y, z$ (in each equivalence class): $xR_{\mathsf{B}}y$ iff $zR_{\mathsf{B}}y$ (in other words, there exists a set of states B all of which elements are R_{B}-accessible by all states in the class, cf. Definition 1). The latter property is a consequence of the fact that R_{B} is a subrelation of R_{U} (by $\mathbf{Inc}_{\mathsf{BU}}$) and that axioms $\mathbf{4}_{\mathsf{BU}}$ and $\mathbf{5}_{\mathsf{BU}}$—which, notice, are Sahlqvist—correspond to the following properties: $\forall x, y, z$ if $xR_{\mathsf{U}}y$ and $yR_{\mathsf{B}}z$ then $xR_{\mathsf{B}}z$; $\forall x, y, z$ if $xR_{\mathsf{U}}y$ and $xR_{\mathsf{B}}z$ then $yR_{\mathsf{B}}z$. Call now this class $\mathfrak{F}$ and consider the class $\mathfrak{F}_g$ of frames in $\mathfrak{F}$ which are point-generated by the equivalence relation R_{U}. Any $\mathcal{F} \in \mathfrak{F}_g$ is thus such that R_{U} is the universal relation on $\mathcal{F}$ and $\mathcal{F}$ contains one unique set B R_{B}-accessible by all elements of the frame. $\mathcal{F}$ is therefore modally equivalent (on the horizontal fragment of $\mathcal{L}^E$) to a $\mathcal{D} \in \mathfrak{D}$. Vice versa, any $\mathcal{D} \in \mathfrak{D}$ is modally equivalent to a $\mathcal{F} \in \mathfrak{F}_g$ (cf. proof of Theorem 1). So $\mathfrak{D}$ is modally equivalent to $\mathfrak{F}_g$ which is, by general results [Blackburn *et al.*, 2001], modally equivalent to $\mathfrak{F}$. Therefore, the axiom system is complete for $\mathfrak{D}$. $\square$

Vertical logic: axiomatics

As to the logic determined by the class of enriched attack graphs on the vertical fragment of $\mathcal{L}^E$ we can proceed in a similar fashion. Notice that an enriched attack graph $\mathcal{E} = \langle A, \to, E\rangle$ can be viewed as a doxastic frame to which a binary relation $\to$ is added. So, consider the logic—call it DA^V—defined by: the rules and axioms of $\mathsf{KD45}$ for modality $\diamondbar_{\mathsf{E}}$, the rules and axioms of K for modality $\diamondbar_{\mathsf{A}}$ and the rules and axioms of $\mathsf{S5}$ for modality $\diamondminus_{\mathsf{U}}$, plus:

$\mathbf{Inc}_{\mathsf{AU}}$	$\diamondbar_{\mathsf{A}}\varphi \to \diamondbar_{\mathsf{U}}\varphi$	$\mathbf{Inc}_{\mathsf{EU}}$	$\diamondbar_{\mathsf{E}}\varphi \to \diamondbar_{\mathsf{U}}\varphi$
$\mathbf{4}_{\mathsf{EU}}$	$\boxbar_{\mathsf{E}}\varphi \to \boxbar_{\mathsf{U}}\boxbar_{\mathsf{E}}\,\varphi$	$\mathbf{5}_{\mathsf{EU}}$	$\diamondbar_{\mathsf{E}}\varphi \to \boxbar_{\mathsf{U}}\diamondbar_{\mathsf{E}}\,\varphi$

Lemma 2. DA^V *is sound and complete for class* $\mathfrak{E}$.

Proof sketch. The proof proceeds as for Lemma 1. □

An axiom system for DA

Everything is now in place to prove the following result:

Theorem 2 (Completeness of DA)**.** *The logic defined by the axioms and rules of* DA^H *and of* DA^V *plus the following instances of* **Com** *and* **Con***:*

$\diamondminus\,\diamondbar\,\varphi \leftrightarrow \diamondbar\,\diamondminus\,\varphi$	$\diamondminus\,\boxbar\,\varphi \to \boxbar\,\diamondminus\,\varphi$

where $\diamondminus \in \{\diamondminus_{\mathsf{B}}, \diamondminus_{\mathsf{U}}\}$*, and* $\diamondbar \in \{\diamondbar_{\mathsf{E}}, \diamondbar_{\mathsf{A}}, \diamondbar_{\mathsf{U}}\}$*, is sound and complete for the class* $\mathfrak{D}\times\mathfrak{E}$ *consisting of products of doxastic frames and enriched attack graphs.*

Proof sketch. The result follows from Lemmata 1 and 2 by [Gabbay and Shehtman, 1998, Theorem 7.12] (cf. our comment regarding completeness of $\mathsf{KD45}\times\mathsf{K}$). □

(Product) Finite model property & decidability

Theorem 3. *Logic* DA *does not have the product finite model property.*

Proof sketch. We exhibit a satisfiable formula from $\mathcal{L}^E$ which cannot be satisfied on a finite product model. Consider the following formula ψ from $\mathcal{L}^E$:

$$\boxbar_{\mathsf{U}}\diamondminus_{\mathsf{B}}\,p \wedge \boxbar_{\mathsf{U}}\boxminus_{\mathsf{B}}\,(p \to \diamondbar_{\mathsf{A}}\neg p) \wedge \boxbar_{\mathsf{U}}\boxminus_{\mathsf{B}}\,(\neg p \to \boxbar_{\mathsf{A}}\neg p)$$

Let $\mathcal{E} = \langle A, E, \to\rangle$ be such that $\langle A, \to\rangle$ consists of an infinite $\leftarrow$-ascending chain of elements $\langle x_n \mid 0 \le n < \omega\rangle$ such that $a_i \not\leftarrow a_j$ for $i < j$. For any choice of E, this is clearly a frame in $\mathfrak{E}$. Let then $\mathcal{D}$ consist of a point s_0 accessing all other states (except itself) in the frame. Take an enumeration $s_0, s_1, \ldots$ of these states. This is clearly a frame in $\mathfrak{D}$. Set $\mathcal{V}(p) = \{(s_{n+1}, a_n) \mid 0 \le n < \omega\}$. We have $\langle\mathcal{D}\times\mathcal{E}, \mathcal{V}\rangle, (s_0, a_0) \models \psi$. Clearly, no model on a finite frame in $\mathfrak{D}\times\mathfrak{E}$ satisfies ψ.[3] □

Theorem 4. *Logic* DA *is decidable.*

Proof sketch. We provide a reduction of DA^H to $\mathsf{S5}$ and of DA^V to K^{U}, proving that the satisfiability of DA is reducible to the satisfiability of $\mathsf{S5}\times\mathsf{K}^{\mathsf{U}}$, which is decidable [Gabbay *et al.*, 2003, Theorem 6.58]. [DA^H to $\mathsf{S5}$]. Take a fresh atom p and define a translation (Boolean clauses omitted): $t(\diamondminus_{\mathsf{U}}\varphi) = \Diamond t(\varphi)$; $t(\diamondminus_{\mathsf{B}}\varphi) = \Diamond(p \wedge t(\varphi))$. We prove: φ is satisfiable in $\mathfrak{D}$ iff $t(\varphi)$ is satisfiable in the class of universal frames. [LEFT TO RIGHT] Assume $\mathcal{M}, x \models \varphi$. Build $\mathcal{M}' = \langle S', \mathcal{V}'\rangle$ with an extra atom p where: $S' = S$ and $\mathcal{V}' = \mathcal{V} \cup \{\langle p, S\rangle\}$. A simple induction shows that: $\mathcal{M}, x \models \varphi$ iff $\mathcal{M}', x \models t(\varphi)$. [RIGHT TO LEFT] Assume $\mathcal{M}, x \models t(\varphi)$. Build $\mathcal{M}' = \langle S', B, \mathcal{V}'\rangle$ where: $S' = S$, $B = \mathcal{V}(p)$ and $\mathcal{V}' = \mathcal{V} - \{\langle p, S\rangle\}$. A simple induction shows that: $\mathcal{M}, x \models t(\varphi)$ iff $\mathcal{M}' \models \varphi$. [$\mathsf{DA}^V$ to K^{U}] Similar. □

[3] The proof goes through also for the weaker logic $\mathsf{KD45}\times\mathsf{K}^{\mathsf{U}}$.

4.3 Argument-based beliefs in DA

This section concludes by showcasing DA as a rich framework for the study of argument-based beliefs.

From beliefs to argument-based beliefs In standard doxastic logic beliefs are properties that are true of all (doxastically) accessible states. In DA, beliefs are properties that are true of all (doxastically) accessible state-argument pairs, *independently of the argument*. So, "I believe that φ" is formalized by $\boxminus_{\mathsf{B}}\,\boxbar_{\mathsf{U}}\,\varphi$ (cf. Remark 3). But now, the argumentative structure available in DA, allows us to differentiate between beliefs based on how they relate to underlying arguments, for instance by being supported by some such arguments. Consider the following formulae (with an attached intuitive reading on the left), ordered from logically weaker to logically stronger (take "$B\varphi$" to stand for "I believe that φ"):

$B\varphi$ with (supporting) arguments	$\boxminus_{\mathsf{B}}(\boxbar_{\mathsf{U}}\varphi \wedge \diamondbar_{\mathsf{U}}\sigma)$
$B\varphi$ with arguments I endorse	$\boxminus_{\mathsf{B}}(\boxbar_{\mathsf{U}}\varphi \wedge \diamondbar_{\mathsf{E}}\sigma)$
$B\varphi$ with arguments which are endorsed and have property ψ	$\boxminus_{\mathsf{B}}(\boxbar_{\mathsf{U}}\varphi \wedge \diamondbar_{\mathsf{E}}(\sigma \wedge \boxminus_{\mathsf{U}}\psi))$

So, the first formula expresses that the agent believes φ and there are arguments for this belief in the sense that each of the doxastic alternatives determining the belief in φ is supported by some argument. The second one refines the first by stressing that arguments for the doxastic alternatives can be found among the arguments the agent endorses. Finally, the third one expresses not only that the beliefs are supported by some endorsed argument, but also that those endorsed arguments all have a given property ψ (notice again the use of $\boxminus_{\mathsf{U}}$ to express that such ψ is a 'row' property, *independent of the doxastic state*). In particular, such ψ can be chosen to express properties such as: "the argument belongs to a given conflict-free set", "the argument belongs to a given stable extension" and the like, to which we now turn.

Dung's argumentation theory in DA We show now how DA can capture some fundamental argumentation-theoretic properties (Table 1). These properties are properties of sets of arguments, that is, properties of row properties (cf. Remark 3). For instance, we want to formalize the property that says that a given row property $\boxminus_{\mathsf{U}}\varphi$ identifies a set of admissible arguments. The strategy for obtaining these formalizations is inspired by [Grossi, 2010]. Let $\blacksquare := \boxminus_{\mathsf{U}}\boxbar_{\mathsf{U}}$:

(i)	φ is a conflict free set	$\blacksquare(\boxminus_{\mathsf{U}}\varphi \to \neg\,\diamondbar_{\mathsf{A}}\,\boxminus_{\mathsf{U}}\varphi)$
(ii)	φ is a self-defended set	$\blacksquare(\boxminus_{\mathsf{U}}\varphi \to \boxbar_{\mathsf{A}}\,\diamondbar_{\mathsf{A}}\,\boxminus_{\mathsf{U}}\varphi)$
(iii)	φ is a fixpoint of $\boxbar_{\mathsf{A}}\diamondbar_{\mathsf{A}}$	$\blacksquare(\boxminus_{\mathsf{U}}\varphi \leftrightarrow \boxbar_{\mathsf{A}}\,\diamondbar_{\mathsf{A}}\,\boxminus_{\mathsf{U}}\varphi)$
(iv)	φ is an admissible set	$(i) \wedge (ii)$
(v)	φ is a complete extension	$(i) \wedge (iii)$
(vi)	φ is a stable extension	$\blacksquare(\boxminus_{\mathsf{U}}\varphi \leftrightarrow \neg\,\diamondbar_{\mathsf{A}}\,\boxminus_{\mathsf{U}}\varphi)$

We comment on (i) and (ii). Formula (i) says that if an argument satisfies row property $\boxminus_{\mathsf{U}}\varphi$, then no attacker of that argument exists which also satisfies the same property. Formula

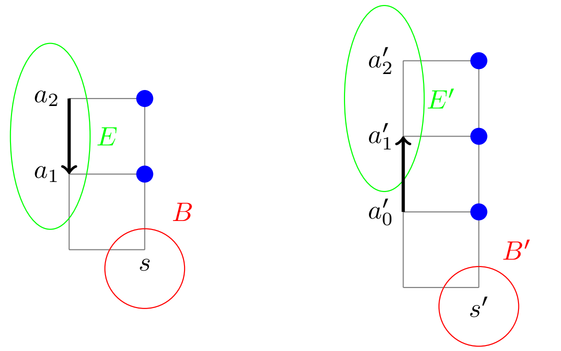

Figure 3: Two frames $\mathcal{F}$ and $\mathcal{F}'$ for the proof of Theorem 5.

(ii) states that all arguments satisfying row property $\boxminus_{\mathsf{U}}\varphi$ are such that all their attackers are defended by some argument satisfying the same property.

Example 5 (After [Modgil, 2009], continued)**.** *Let us go back to the model on the right of Figure 2. It is a validity of the model that: the agent believes that* s *is the case; that it has arguments for that belief, which it endorses; and that those arguments also have the property of belonging to the conflict-free set of arguments specified by property* BBC*:*

$$\boxminus_{\mathsf{B}}(\boxbar_{\mathsf{U}}\mathsf{s} \wedge \diamondbar_{\mathsf{E}}(\sigma \wedge \boxminus_{\mathsf{U}}(\mathsf{BBC} \wedge (\blacksquare(\boxminus_{\mathsf{U}}\mathsf{BBC} \rightarrow \neg \diamondbar_{\mathsf{A}} \boxminus_{\mathsf{U}}\mathsf{BBC})))))$$

It is also a validity of the model that set of arguments supporting the agent's doxastic alternatives is an admissible set:

$$\blacksquare(\diamondminus_{\mathsf{B}}\sigma \rightarrow \neg \diamondbar_{\mathsf{A}} \diamondminus_{\mathsf{B}}\sigma) \wedge \blacksquare(\diamondminus_{\mathsf{B}}\sigma \rightarrow \boxbar_{\mathsf{A}} \diamondbar_{\mathsf{A}} \diamondminus_{\mathsf{B}}\sigma)$$

Notice that $\diamondminus_{\mathsf{B}}\sigma$ *is a row property* ($\diamondminus_{\mathsf{B}}\sigma \leftrightarrow \boxminus_{\mathsf{U}} \diamondminus_{\mathsf{B}} \sigma$)*.*

Argumentative properties of endorsed arguments We conclude with a few technical considerations about the expressivity of DA. The previous subsection has shown how basic notions from argumentation theory can be characterized at the level of models in DA. We now look at the feasibility of characterizations at frame level: are there formulae which characterize whether the set of endorsed arguments E is conflict-free and self-defended? This is a novel application of frame correspondence theory [van Benthem, 1983] to abstract argumentation. We start by the following:

Proposition 2. *Let* $\mathcal{F} = \mathcal{D} \times \mathcal{E}$*. If* $\mathcal{F} \models \boxbar_{\mathsf{E}}(\boxminus_{\mathsf{U}}p \rightarrow \neg \diamondbar_{\mathsf{A}} \boxminus_{\mathsf{U}}p)$ *then* E *in* $\mathcal{E}$ *is conflict free.*

We next show that Proposition 2 cannot be strengthened to a *characterization* of conflict-freeness:

Theorem 5. *There exists no formula* φ *of* DA *s.t.:* $\mathcal{F} \models \varphi$ *iff* E *is conflict-free, for* $\mathcal{F} \in \mathfrak{D} \times \mathfrak{E}$*.*

Proof sketch. Consider the frames $\mathcal{F}$ and $\mathcal{F}'$ of Figure 3. We claim that $\forall \psi \in \mathcal{L}^E$ $\mathcal{F}' \models \psi \Rightarrow \mathcal{F} \models \psi$, while at the same time, E' in $\mathcal{F}'$ is conflict free. □

However, a characterization *does* exists for self-defense:

Proposition 3 (Characterization of self-defense of E)**.** *Let* $\mathcal{F} = \mathcal{D} \times \mathcal{E}$*. If* $\mathcal{F} \models (\boxbar_{\mathsf{E}}\, p \wedge \diamondbar_{\mathsf{E}} \diamondbar_{\mathsf{A}}\, q) \rightarrow \diamondbar_{\mathsf{U}}(q \wedge \diamondbar_{\mathsf{A}}p)$ *if and only if* E *in* $\mathcal{E}$ *is self-defended.*

5 Conclusions

We proposed an approach based on product logics to study the interaction between the information state of an agent and the arguments the agent endorses. The approach has been illustrated by two product logics, which have been studied from a logical point of view, and have been used to model simple scenarios and formalize a number of interaction properties between beliefs and arguments.

The paper has started from the simplest possible doxastic logic, KD45. We see several natural directions to extend the work: by covering the multi-agent case; by incorporating richer order-theoretic models for doxastic logic [Baltag and Smets, 2008]; by studying the dyanmics supported by both the horizontal and vertical dimensions with tools from dynamic epistemic logic [van Ditmarsch *et al.*, 2007].

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